\textbf{Abstract}

A singular perturbed variational problem is considered. A method for providing lower bounds to the energy is shown. The lower bound will be found to be the leading order $\Gamma$-limit as our small parameter, $\epsilon$, goes to zero.
Outline

1. Introduction and some motivation.

2. Introduction to $\Gamma$-convergence.

3. Lower bounds as the first order $\Gamma$-limit.

Introducing the Energy Functional

• Singular perturbed variational problem

\[
E := \min \ E(\nabla u) = \int_{\Omega} \epsilon^{-1}(1 - |\nabla u|^2)^2 + \epsilon |\nabla \nabla u|^2, \\
\Omega \in \mathbb{R}^2, \\
|\nabla \nabla u|^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2.
\]

• Interested in asymptotic behavior as \( \epsilon \to 0 \).
  \( \Rightarrow \) sharp lower bounds for \( E(\nabla u) \).
Physical Motivation

1. Landau theory in application to liquid crystals.
   - Minimizing "some" energy yields governing equations for the director field $\frac{\nabla u}{|\nabla u|}$.

2. Thin films.
   - Modeling blisters in compressed thin films.
   - Fold patterns of blisters is seen as minimizing the energy between the sum of the membrane and the bending energy.

3. Ferromagnetism, etc.
Yet Another Example

- Gradient theory in phase transitions.
  - Minimizing an energy, subject to a mass constraint
    \[
    \min\{ E(v) = \int_\Omega W(v) dx \mid v : \Omega \to [0, 1], \int_\Omega v dx = C \}. 
    \]
  - \( W(v) \) is some double well potential with local minima at \( v = \alpha, \beta \).
  - No information given about the transition from \( \alpha \) to \( \beta \).
  - Add a singular gradient term, i.e.
    \[
    \min\{ \int_\Omega W(v) + \epsilon^2 |\nabla u|^2 dx \mid v : \Omega \to [0, 1], \int_\Omega v dx = C \}. 
    \]
Yet Another Example (cont.)

- Perturbation yields a solution of

\[ u^\epsilon \approx u(x) + u_1(\frac{x}{\epsilon}), \]
\[ u(x) : \Omega \rightarrow \{\alpha, \beta\}, \]
\[ u_1(x) : \mathbb{R} \rightarrow \mathbb{R}, \quad u_1(x \rightarrow \pm\infty) = 0. \]

- For each piece to have the same relative size as \( \epsilon \rightarrow 0 \) consider

\[
\min \{ \int_\Omega \epsilon^{-1} W(v) + \epsilon |\nabla u|^2 dx \mid v : \Omega \rightarrow [0, 1], \int_\Omega v dx = C \}. \quad (2)
\]
Aside on $\Gamma$-convergence

**Definition:** ($\Gamma$-convergence). We say that a sequence of function $f_j : X \to \bar{\mathbb{R}}$ $\Gamma$-converges in $X$ to $f_\infty : X \to \bar{\mathbb{R}}$ if for all $x \in X$ we have

- (i) (lim inf inequality) for every sequence $x_j$ converging to $x$

  $$f_\infty(x) \leq \liminf_j f_j(x_j).$$

- (ii) (lim sup inequality) there exists a sequence $x_j$ converging to $x$ such that

  $$f_\infty(x) \geq \limsup_j f_j(x_j).$$

The function $f_\infty$ is called the $\Gamma$-limit of $f_j$ and write $f_\infty = \Gamma\text{-}\lim_j f_j$.

**Note:**

$$f_\infty(x) = \min\{\liminf_j f_j(x_j) : x_j \to x\} = \min\{\limsup_j f_j(x_j) : x_j \to x\}.$$ 

**Another Note:**

$\Gamma$-convergence gives an avenue to lower bounds.
An Example on the Real Line

• Consider the following sequence of functions

\[ f_j(x) = \begin{cases} 
-1 & \text{if } jt = 1 \\
0 & \text{otherwise}
\end{cases}. \]

• The pointwise limit:

\[ f_j \to 0. \]

• The \( \Gamma \)-limit:

\[ \Gamma \text{-lim}_j f_j = \begin{cases} 
0 & \text{if } t \neq 0 \\
-1 & \text{if } t = 0
\end{cases}. \]
\( \Gamma\)-limit: a First Step

- General functional form:

\[
E_\epsilon(v) = \int F(v) = \int \epsilon^{-1} W(v) + \epsilon g^2(\nabla v),
\]

\[
W \geq 0,
\]

\[
W(v) = 0 \text{ for } v = a, b.
\]
\( \Gamma \)-limit: a First Step (cont.)

- Barroso and Fonseca (1994) showed:

\[
\Gamma \lim_{\epsilon \to 0} E_\epsilon = \int_D K(\nu)ds,
\]

- \( D \) is the jump (defect) set of \( \nu \) and \( \nu \) is the normal direction of \( D \).

- \( K(\nu) \) is local functional problem of the form

\[
K(\nu) = \lim_{\epsilon \to 0} \inf_{v \in A(\nu)} \int_{Q_\nu} F(v)
\]

\[
A(\nu) = \{ v \mid v(y) = a \text{ if } y \cdot \nu \leq -\frac{1}{2}, \quad v(y) = b \text{ if } y \cdot \nu \leq \frac{1}{2}, \quad u \text{ is periodic with period 1 in the directions of } \nu_1, \ldots, \nu_{N-1}. \}
\]

- \( K(\nu) \) seeks the optimal profile of a jump from \( a \) to \( b \) across a plane normal to \( \nu \) with its resulting energy.
A Similar Setup

- \( \Gamma \)-limit still in the form over some defect set, \( D \).

- Jin and Kohn (1999) proposed the resulting local problem is

\[
K_\epsilon = \min_{u \in A} \int_\Omega F(\nabla u) = \min_{u \in A} \int_\Omega \epsilon^{-1}(1 - |\nabla u|^2)^2 + \epsilon|\nabla \nabla u|^2,
\]

\[
A = \{ u \mid \nabla u(x, \pm \frac{1}{2}) = (a, \mp \sqrt{1 - a^2}) \text{, } \nabla u \text{ is periodic with period 1 in the x-direction.} \}
\]

- \( \Omega \) is the unit square in \( \mathbb{R}^2 \).
**Goal:** Find a function $\phi(\nabla u)$ whose divergence is dominated by $K_\epsilon$.

- Many such functions.

- Want one that just works!

\[
\phi(\nabla u) = 2 \left( -\frac{1}{3}u_x^3 - u_xu_y^2 + u_x, \quad \frac{1}{3}u_y^3 + u_yu_x^2 - u_y \right)
\]

- By Cauchy’s inequality,

\[
\text{div}(\phi(\nabla u)) \leq F(\nabla u) - 2\epsilon(u_{xx}u_{yy} - u_{xy}^2).
\]
Bounding $K_\epsilon$ (cont.)

- Integrating over $\Omega$, using the divergence theorem, and integration by parts,

$$
\int_{\partial \Omega} \phi(\nabla u) \cdot n + 2\epsilon \int_{\partial \Omega} u_x du_y \leq \int_{\Omega} F(\nabla u).
$$

- This is applicable for any functional with sufficient smoothness!

- For the above example, applying the boundary conditions,

$$
\frac{8}{3} \sqrt{1 - a^2} \leq \int_{\Omega} \epsilon^{-1}(1 - |\nabla u|^2)^2 + \epsilon |\nabla \nabla u|^2.
$$
Conclusions

1. A method for yielding asymptotically sharp lower bounds was shown and was motivated through $\Gamma$-convergence.

2. The lower bound is only dependent on the boundary data and thus sharp lower bounds are easily obtained for many differing boundary conditions.

3. The freedom of choosing an appropriate $\phi(\nabla u)$ will be noted in the speed of converging to the $\Gamma$-limit.
References


