1. Define the set $R[[x]]$ of formal power series in the indeterminate $x$ with coefficients from $R$ to be all formal infinite sums

$$
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.
$$

Define addition and multiplication of power series in the same way as for power series with real or complex coefficients; i.e., extend polynomial addition and multiplication to power series as though they were “polynomials of infinite degree:”

$$
\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n
$$

and

$$
\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) x^n.
$$

(a) (3 points) Prove that $R[[x]]$ is a commutative ring with 1.

Solution:

(b) (3 points) Show that $1 - x$ is a unit in $R[[x]]$ with inverse $1 + x + x^2 + \cdots$.

Solution:

(c) (3 points) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if $a_0$ is a unit in $R$.

Solution:
2. (5 points) Prove that if $R$ is an integral domain then the ring of formal power series $R[[x]]$ is also an integral domain.

Solution:

3. (5 points) Prove that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic even though they are isomorphic as abelian groups.

Solution:

4. (5 points) Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Solution:

5. (6 points) Decide which of the following are ideals of the ring $\mathbb{Z}[x]$. Be sure to justify your answer.

(a) the set of all polynomials whose constant term is a multiple of 3

Solution:

(b) the set of all polynomials whose coefficient of $x^2$ is a multiple of 3

Solution:

(c) the set of all polynomials whose constant term, the coefficient of $x$, and the coefficient of $x^2$ are zero

Solution:

(d) $\mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of $x$ occur)

Solution:

(e) the set of all polynomials whose coefficients sum to zero

Solution:

(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to $x$. 


6. Let $D$ be an integer that is not a perfect square in $\mathbb{Z}$ and let

$$S = \left\{ \begin{pmatrix} a & b \\Db & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$ 

(a) (3 points) Prove that $S$ is a subring of $M_2(\mathbb{Z})$.

Solution:

(b) (3 points) If $D$ is not a perfect square in $\mathbb{Z}$, prove that the map $\varphi : \mathbb{Z}[\sqrt{D}] \to S$ defined by

$$\varphi(a + b\sqrt{D}) = \begin{pmatrix} a & b \\Db & a \end{pmatrix}$$

is a ring isomorphism.

Solution:

(c) (3 points) If $D \equiv 1 \mod 4$ is square free, prove that the set

$$\left\{ \begin{pmatrix} a & b \\ (D-1)b/4 & a + b \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$$

is a subring of $M_2(\mathbb{Z})$ and is isomorphic to the quadratic integer ring $\mathcal{O}$.

Solution:

7. Let $I$ and $J$ be ideals of $R$.

(a) (3 points) Prove that $I + J$ is the smallest ideal of $R$ containing both $I$ and $J$.

Solution:

(b) (3 points) Prove that $IJ$ is an ideal contained in $I \cap J$.

Solution:

(c) (3 points) Give an example where $IJ \neq I \cap J$.

Solution:

(d) (3 points) Prove that if $R$ is commutative and if $I + J = R$, then $IJ = I \cap J$. 

Solution:
Solution: