1. (3 points) Assume that $R$ is a commutative ring with identity. Prove that $R$ is a field if and only if $0$ is a maximal ideal.

**Solution:**

2. (3 points) Assume that $R$ is a commutative ring with identity. Prove that if $P$ is a prime ideal of $R$ and $P$ contains no zero divisors then $R$ is an integral domain.

**Solution:**

3. Let $x^2 + x + 1$ be an element of the polynomial ring $E = \mathbb{F}_2[x]$ and use the bar notation to denote the passage to the quotient ring $\mathbb{F}_2[x]/(x^2 + x + 1)$.

   (a) (3 points) Prove that $\bar{E}$ has 4 elements: $\bar{0}$, $\bar{1}$, $\bar{x}$, and $\bar{x+1}$.

   **Solution:**

   (b) (3 points) Write out the $4 \times 4$ addition table for $\bar{E}$ and reduce that the additive groups $\bar{E}$ is isomorphic to the Klein 4-group.

   **Solution:**

   (c) (3 points) Write out the $4 \times 4$ multiplication table for $\bar{E}$ and prove that $\bar{E}^\times$ is isomorphic to the cyclic group of order 3. Deduce that $\bar{E}$ is a field.
4. (3 points) Let $R$ be a finite commutative ring with identity. Prove that every prime ideal of $R$ is a maximal ideal.

Solution:

5. (5 points) Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

Solution: