1. (5 points) Use the submodule criterion to show that kernels and images of $R$-module homomorphisms are submodules.

Solution:

2. (5 points) Let $A$ be any $\mathbb{Z}$-module, let $a$ be an element of $A$ and let $n$ be a positive integer. Prove that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi_a(\overline{k}) = ka$ is a well-defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A \mid na = 0\}$ (so $A_n$ is the annihilator in $A$ of the ideal $(n)$ of $\mathbb{Z}$—cf. Exercise 10, Section 1).

Solution:

3. (5 points) Find all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$.

Solution:

4. (5 points) Prove that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.

Solution: