The William Lowell Putnam Mathematical Competition

October 25, 2014

1. Given a positive integer \( n \), what is the largest \( k \) such that the numbers \( 1, 2, \ldots, n \) can be put into \( k \) boxes so that the sum of the numbers in each box is the same? [When \( n = 8 \), the example \( \{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\} \) shows that the largest \( k \) is at least 3.]

Solution: The largest such \( k \) is \( \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \). For \( n \) even, this value is achieved by the partition

\[
\{1, n\}, \{2, n-1\}, \ldots;
\]

for \( n \) odd, it is achieved by the partition

\[
\{n\}, \{1, n-1\}, \{2, n-2\}, \ldots.
\]

One way to see that this is optimal is to note that the common sum can never be less than \( n \), since \( n \) itself belongs to one of the boxes. This implies that \( k \leq (1 + \cdots + n)/n = (n + 1)/2 \). Another argument is that if \( k > (n + 1)/2 \), then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.

2. Find all differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
f'(x) = \frac{f(x+n) - f(x)}{n}
\]

for all real numbers \( x \) and all positive integers \( n \).

Solution: The only such functions are those of the form \( f(x) = cx + d \) for some real numbers \( c, d \) (for which the property is obviously satisfied). To see this, suppose that \( f \)
has the desired property. Then for any \( x \in \mathbb{R} \),
\[
2f'(x) = f(x + 2) - f(x)
\]
\[
= (f(x + 2) - f(x + 1)) + (f(x + 1) - f(x))
\]
\[
= f'(x + 1) + f'(x).
\]
Consequently, \( f'(x + 1) = f'(x) \).

Define the function \( g : \mathbb{R} \to \mathbb{R} \) by \( g(x) = f(x + 1) - f(x) \), and put \( c = g(0), d = f(0) \).
For all \( x \in \mathbb{R} \), \( g'(x) = f'(x + 1) - f'(x) = 0 \), so \( g(x) = c \) identically, and \( f'(x) = f(x + 1) - f(x) = g(x) = c \), so \( f(x) = cx + d \) identically as desired.