1. A square grid on the Euclidean plane consists of all points \((m,n)\), where \(m\) and \(n\) are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

**Solution:** It is not possible. The proof is by contradiction. Suppose that such a covering family \(\mathcal{F}\) exists. Let \(D(P,\rho)\) denote the disc with center \(P\) and radius \(\rho\). Start with an arbitrary disc \(D(O,r)\) that does not overlap any member of \(\mathcal{F}\). Then \(D(O,r)\) covers no grid point. Take the disc \(D(O,r)\) to be maximal in the sense that any further enlargement would cause it to violate the non-overlap condition. Then \(D(O,r)\) is tangent to at least three discs in \(\mathcal{F}\). Observe that there must be two of the three tangent discs, say \(D(A,a)\) and \(D(B,b)\), such that \(\angle AOB \leq 120^\circ\). By the Law of Cosines applied to triangle \(ABO\),

\[
(a + b)^2 \leq (a + r)^2 + (b + r)^2 + (a + r)(b + r),
\]

which yields

\[
ab \leq 3(a + b)r + 3r^2, \quad \text{and thus} \quad 12r^2 \geq (a - 3r)(b - 3r).
\]

Note that \(r < 1/\sqrt{2}\) because \(D(O,r)\) covers no grid point, and \((a-3r)(b-3r) \geq (5-3r)^2\) because each disc in \(\mathcal{F}\) has radius at least 5. Hence \(2\sqrt{3}r \geq (5-3r)\), which gives \(5 \leq (3 + 2\sqrt{3})r < (3 + 2\sqrt{3})/\sqrt{2}\) and thus \(5\sqrt{2} < 3 + 2\sqrt{3}\). Squaring both sides of this inequality yields \(50 < 21 + 12\sqrt{3} < 21 + 12 \cdot 2 = 45\). This contradiction completes the proof.

2. Let \(ABCD\) be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60°. Prove that

\[
\frac{1}{3}|AB^3 - AD^3| \leq |BC^3 - CD^3| \leq 3|AB^3 - AD^3|.
\]

When does equality hold?
Solution: By symmetry, we only need to prove the first inequality.
Because quadrilateral $ABCD$ has an incircle, we have $AB + CD = BC + AD$, or $AB - AD = BC - CD$. It suffices to prove that
\[
\frac{1}{3}(AB^2 + AB \cdot AD + AD^2) \leq BC^2 + BC \cdot CD + CD^2.
\]
By the given condition, $60^\circ \leq \angle A, \angle C \leq 120^\circ$, and so $\frac{1}{2} \geq \cos A, \cos C \geq -\frac{1}{2}$. Applying the law of cosines to triangle $ABD$ yields
\[
BD^2 = AB^2 - 2AB \cdot AD \cos A + AD^2 \geq AB^2 - AB \cdot AD + AD^2 \\
\geq \frac{1}{3}(AB^2 + AB \cdot AD + AD^2).
\]
The last inequality is equivalent to the inequality $3AB^2 - 3AB \cdot AD + 3AD^2 \geq AB^2 + AB \cdot AD + AD^2$, or $AB^2 - 2AB \cdot AD + AD^2 \geq 0$, which is evident. The last equality holds if and only if $AB = AD$.

On the other hand, applying the Law of Cosines to triangle $BCD$ yields
\[
BD^2 = BC^2 - 2BC \cdot CD \cos C + CD^2 \leq BC^2 + BC \cdot CD + CD^2.
\]
Combining the last two inequalities gives the desired result.

For the given inequalities to hold, we must have $AB = AD$. This condition is also sufficient, because all the entries in the equalities are 0. Thus, the given inequalities hold if and only if $ABCD$ is a kite with $AB = AD$ and $BC = CD$. 