Graphing (non-calculator) Portion of Exam

1. Find the x- and y-intercepts of the graph of the equation. \(2y - xy + 3x = 4\)
   - x-int: 
     \[3x = 4\]
     \[x = \frac{4}{3}\]
     \[\left(\frac{4}{3}, 0\right)\]
   - y-int:
     \[2y \cdot (0)y + 3(0) = 4\]
     \[y = \frac{4}{3}\]
     \[(0, \frac{4}{3})\]

2. Solve using the quadratic formula: \(6x^2 + 5 = 40x - 10x^2\)
   \[-4x^2 - 40x + 5 = 0\]
   \[a = -4\]
   \[b = -40\]
   \[c = 5\]
   \[x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(-4)(5)}}{2(-4)}\]
   \[x = \frac{40 \pm \sqrt{1600 - 80}}{8}\]
   \[x = \frac{40 \pm 1160}{8}\]
   \[x = \frac{1160 - 8}{8}\]
   \[x = 112\]
   \[\left(\frac{112}{8}, \frac{40}{8}\right)\]
   \[\left(\frac{14}{1}, \frac{5}{1}\right)\]

3. Solve for \(x\):
   - a. \(\left(\sqrt{x + 1}\right)^2 = 3x + 1\)
     \[x + 1 = (3x + 1)(3x + 1)\]
     \[x + 1 = 9x^2 + 6x + 1\]
     \[0 = 9x^2 + 6x\]
     \[0 = x(9x + 5)\]
     \[9x + 5 = 0\]
     \[x = -\frac{5}{9}\]
   - b. \(4x^2 + 4x = 7\)
     \[4x^2 + 4x - 7 = 0\]
     \[a = 4\]
     \[b = 4\]
     \[c = -7\]
     \[x = \frac{-4 \pm \sqrt{4^2 - 4(4)(-7)}}{2(4)}\]
     \[x = \frac{-4 \pm \sqrt{16 + 112}}{8}\]
     \[x = \frac{-4 \pm 12}{8}\]
     \[x = \frac{-4 \pm 12}{8}\]
     \[x = \frac{-4 + 12}{8}\]
     \[x = \frac{8}{8}\]
     \[x = 1\]
   - c. \(\frac{12x + 5}{4x + 5} = 3\)
     \[3(12x + 5) = 4x + 15\]
     \[36x + 15 = 4x + 15\]
     \[30x = 4x\]
     \[32x = 0\]
     \[x = 0\]
   - d. \(4\sqrt{x} - 3 = 0\)
     \[4\sqrt{x} = 3\]
     \[(\sqrt{x})^2 = (3/4)^2\]
     \[x = \frac{9}{16}\]

4. Consider the points \((-6, 4)\) and \((-2, 3)\).
   - a. Find the slope of the line passing through these points.
     \[m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{-4} = -\frac{1}{4}\]
   - b. Find the equation of the line passing through these points. (Put answer in slope-intercept form.)
     \[y - y_1 = m(x - x_1)\]
     \[y - 4 = -\frac{1}{4}(x + 6)\]
     \[y - 4 = -\frac{1}{4}x - 3\]
     \[y = -\frac{1}{4}x - 3 + 4\]
     \[y = -\frac{1}{4}x + 1\]
5. Let \( f(x) = \sqrt{x + 9} \), \( g(x) = x^2 + 2 \) and \( h(x) = 2x - 1 \). Find the following:
   a. \( f(0) \)
      \[
      f(0) = \sqrt{0 + 9} = \sqrt{9} = 3
      \]
   b. \( g(x+2) \)
      \[
      g(x+2) = (x+2)^2 + 2 = x^2 + 4x + 4 + 2 = x^2 + 4x + 6
      \]
   c. \( (h-g)(x) = h(x) - g(x) \)
      \[
      (h-g)(x) = (2x-1) - (x^2 + 2) = 2x - 1 - x^2 - 2 = -x^2 + 2x - 3
      \]
   d. \( (g \circ f)(x) = g(f(x)) \)
      \[
      (g \circ f)(x) = g(\sqrt{x + 9}) = (\sqrt{x + 9})^2 + 2 = x + 9 + 2 = x + 11
      \]
   e. \( (h \circ g)(-2) = h(g(-2)) = h((2)^2 + 2) = h(6) = 2(6) - 1 = 11
      \]

6. Let \( f(x) = 3x - 2 \) and \( g(x) = x^2 + 3 \). Find the following:
   a. \( (f-g)(2t) \)
      \[
      (f-g)(2t) = f(2t) - g(2t) = 3(2t) - 2 - ((2t)^2 + 3) = 6t - 2 - (4t^2 + 3) = -4t^2 + 6t - 5
      \]
   b. \( (f \circ g)(x) = f(g(x)) \)
      \[
      (f \circ g)(x) = f(x^2 + 3) = 3(x^2 + 3) - 2 = 3x^2 + 9 - 2 = 3x^2 + 7
      \]

7. Find the vertex of the parabola \( f(x) = -3x^2 + 12x - 20. \)
   \[
   x = \frac{-b}{2a} = \frac{-12}{2(-3)} = \frac{-12}{-6} = 2
   \]
   \[
   f(2) = -3(2)^2 + 12(2) - 20 = -12 + 24 - 20 = -8
   \]
   \[
   \text{vertex: } (2, -8)
   \]

8. State the domains of the following functions in interval notation.
   Example: \( f(x) = \sqrt{x + 2} \) \hspace{1cm} \text{Domain: } [-2, \infty)

   a. \( f(x) = 3\log_4(x) \) \hspace{1cm} \text{Domain: } (0, \infty)

   b. \( f(x) = 3e^{2x} \) \hspace{1cm} \text{Domain: } (-\infty, \infty)

   c. \( f(x) = \frac{x^2}{x-8} \) all real's except 8 \hspace{1cm} \text{Domain: } (-\infty, 8) \cup (8, \infty)

9. Consider the following graph: (if none exist, write none)
   a. On what intervals (if any) is the graph increasing? \((-\infty, 2] \]
      decreasing? \([2, \infty) \]
      constant? \(\text{none} \]
   b. Is \( y \) a function of \( x \)? \textbf{Explain your reasoning.}
      \(\text{yes, passes vertical line test}\)
c. Does this graph have an inverse function? **Explain your reasoning.**  
**No**, does not pass **Horizontal Line Test**

d. What is the domain of this function?  
\((-\infty, \infty)\)

e. What is the range of this function?  
\([-2, 0]\)

10. Consider the following function  
\[ f(x) = -2x^9 + 4x^3 - 17 \]

a. What is the maximum number of zeros?  
9

b. What is the maximum number of turning points?  
8

c. Draw an end behavior diagram for the function.

11. Consider the graph of  
\[ f(x) = x^2 - 5x \]

Use your knowledge of transformations to write an equation for the following description:  
\[
\begin{align*}
(+4 \ outside) & \quad (-3 \ inside) \\
(\text{+4 outside}) & \quad (-\text{3 inside})
\end{align*}
\]

The graph of \( f \) is shifted four units up, reflected about the y-axis, and vertically stretched by a factor of 3.

\[
g(x) = 3f(-x) + 4 = 3((-x)^2 - 5(-x)) + 4
\]

\[
= 3(x^2 + 5x) + 4
\]

\[
= 3x^2 + 15x + 4
\]

12. Let  
\[ g(x) = -x^2 + 4 \]

Find the following (if they do not occur, write "none") and sketch the graph of \( g \) on the provided coordinate plane.

a. Describe the sequence of transformations from
\[ f(x) = x^2 \text{ to } g. \]
reflect over x-axis  
shift up 4

b. Identify any interval(s) of the domain for which the function is increasing.

\[ (-\infty, 0] \]

c. Identify any interval(s) of the domain for which the function is decreasing.

\[ [0, \infty) \]

13. For the following rational function, give all vertical, horizontal, and slant asymptotes of the graph of the function (if any such asymptotes exist).

\[ f(x) = \frac{3x^2 + 21x + 35}{2x^2 + 18x + 40} \]

**VA:**  
\[
\frac{2x^2 + 18x + 40}{a} = 0
\]

\[
x^2 + 9x + 20 = 0
\]

\[
(x + 4)(x + 5) = 0
\]

\[
x + 4 = 0 \quad x + 5 = 0
\]

\[
x = -4 \quad x = -5
\]

**HA:**  
\[ n = 2 \quad m = 2 \quad n = m \]

\[ y = \frac{a}{b} = \frac{3}{a} \]

\[ y = \frac{3}{a} \]

**OA:** none
14. For the function \( g(x) = \frac{x^2 + 2x + 1}{x - 4} \) find the following asymptotes. Write each asymptote as an equation.
   a. All horizontal asymptotes (if any exist)
      \[ n = 2, \quad m = 1 \]
      \[ n > m \]
      \[ \text{no HA} \]
   b. All vertical asymptotes (if any exist)
      \[ x - 4 = 0 \]
      \[ x = 4 \]
   c. All slant asymptotes (if any exist)
      \[ y = x + 6 \]

15. Find the inverse of the function \( f(x) = -3 + \frac{3}{x + 8} \)
   \[ x = -3 + \frac{3}{y + 8} \]
   \[ (x + 3)^3 = \left(\frac{3}{y + 8}\right)^3 \]
   \[ (x + 3)^3 = y + 8 \]
   \[ x + 3 = \sqrt[3]{y + 8} \]
   \[ x = \sqrt[3]{y + 8} - 3 \]
   \[ f^{-1}(x) = (x + 3)^3 - 8 \]

16. Find the inverse function of the following, if they exist. If the inverse function does not exist, state why
   a. \( f(x) = \frac{3x + 4}{5} \)
      \[ x = \frac{3y + 4}{5} \]
      \[ 5x = 3y + 4 \]
      \[ 5x - 4 = 3y \]
      \[ \frac{5x - 4}{3} = y \]
      \[ f^{-1}(x) = \frac{5x - 4}{3} \]
   b. \( f(x) = (x - 5)^2 \)
      \[ x = \sqrt{2y + 3} \]
      \[ x^2 = 2y + 3 \]
      \[ x^2 - 3 = 2y \]
      \[ y = \frac{x^2 - 3}{2} \]
      \[ f^{-1}(x) = \frac{x^2 - 3}{2}, \quad x \geq 0 \]
   c. \( f(x) = \sqrt{2x + 3} \)
      \[ x = \sqrt{2y + 3} \]
      \[ x^2 = 2y + 3 \]
      \[ x^2 - 3 = 2y \]
      \[ y = \frac{x^2 - 3}{2} \]

17. a. Compute: \( x^3 - 4x^2 + 3x + 18 \) divided by \( x - 3 \).
   \[ \begin{array}{c|cccc}
   3 & 1 & -4 & -3 & 18 \\
   & 3 & -3 & -18 & 0 \\
   \hline
   & 1 & -1 & -6 & \end{array} \]
   \[ x^2 - x - 6 \]
   b. Using your answer from part a, find all the real zeros of the function algebraically. Be sure to include the multiplicity of repeated zeros.
   \[ f(x) = x^3 - 4x^2 - 3x + 18 \]
   \[ f(x) = (x - 3)(x^2 - x - 6) \]
   \[ = (x - 3)(x - 3)(x + 2) \]
   \[ x - 3 = 0 \quad x + 2 = 0 \]
   \[ x = 3, \quad x = -2 \]
   \[ \text{mult. 2} \]
18. Consider the following function: \( f(x) = x^3 - 3x^2 - 24x - 28 \)

a. Verify that \((x+2)\) is a factor of \( f(x)\)

\[
\begin{array}{c|cccc}
-2 & 1 & -3 & -24 & -28 \\
  & 1 & 10 & 28 & 0 \\
\end{array}
\]

\( x+2 \) is a factor of \( f(x) \)

Since the remainder is zero

b. Find all of the zeros of \( f(x) \) and include their multiplicity.

\[
f(x) = (x+2)(x^2 - 5x - 14) = (x+2)(x+2)(x-7)
\]

\( x = -2, x = 7 \) \( \text{w/ mult. of 2} \)

19. Use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms. (Assume all variables represent positive numbers.)

\[
\log_a \left( \frac{\sqrt{y}}{z^2} \right) = \log_a \sqrt{y} - \log_a z^2
\]

\[
= \frac{1}{2} \log_a y - 2 \log_a z
\]

20. Condense the expression \( 2 \log_{10} (t+1) - \log_{10} z + \log_{10} 2 \) as much as possible.

\[
alog_a \left( \frac{\sqrt{y}}{z^2} \right) = \log_a \sqrt{y} - \log_a z
\]

\[
= \frac{1}{2} \log_a y - 2 \log_a z
\]

21. Expand the expression \( \log_{10} \left( \frac{x \cdot y}{z} \right) \) as much as possible.

\[
= \log_{10} x + \log_{10} y - \log_{10} z
\]

22. Given \( f(x) = \log_5 (x+5) \), find the following. If any do not occur, write “None”.

a. Domain: \(( -3, \infty )\)

b. Range: \(( -\infty, \infty )\)

c. Vertical Asymptote: \( x = -3 \)

d. Horizontal Asymptote: none

e. \( x \)-intercept(s): \((-2, 0)\)

f. Sketch graph of \( f \), including asymptotes
23. Solve the following for \( x \).
   a. \( 4 = \log_2(x - 5) \)
      \[ 2^4 = 2^{\log_2(x-5)} \]
      \[ 16 = x - 5 \]
      \[ 21 = x \]
   b. \( \log_2 x + \log_2(x + 2) = \log_2(x + 6) \)
      \[ \log_2 x(x+2) = \log_2(x+6) \]
      \[ x(x+2) = x+6 \]
      \[ x^2 + 2x = x + 6 \]
      \[ x^2 + x - 6 = 0 \]
      \[ (x+3)(x-2) = 0 \]
      \[ x = -3 \]
      \[ x = 2 \]

24. Solve the following system of equations using either substitution or elimination. (Choose one of the methods and show your work!)
   \[
   \begin{align*}
   2(3x+2y &= 3) \\
   -3(2x-3y &= 15) \\
   3x + 2(-3) &= 3 \\
   3x - 6 &= 3 \\
   3x &= 9 \\
   x &= 3 \\
   \end{align*}
   \]
   \[ \begin{align*}
   6x + 4y &= 60 \\
   -10x + 9y &= -45 \\
   13y &= -39 \\
   y &= -3 \\
   \end{align*} \]
   \[ (3, -3) \]

25. Solve for \( x \) and \( y \) in the following system. If the system does not have a solution or has infinitely many solutions indicate so.
   \[ 2 \left( -\frac{1}{2}x - 3y = 1 \right) \]
   \[ x + 6y = 0 \]
   \[ \begin{align*}
   -x - 6y &= 2 \\
   x + 6y &= 0 \\
   0 \neq 2 & \text{ NO SOLUTION} \\
   \end{align*} \]

26. Solve the following system of equations. If the system does not have a solution or has infinitely many solutions indicate so.
   \[
   \begin{align*}
   x - y + 3z &= 1 \\
   2(2y - z &= 2) \\
   -3y + 2z &= 0 \\
   \end{align*}
   \]
   \[ \begin{align*}
   4y - 2z &= y \\
   -3y + 2z &= 0 \\
   \end{align*} \]
   \[ y = 4 \]

\[ \begin{align*}
   x - 4 + 3(6) &= 1 \\
   x - 4 + 18 &= 1 \\
   x + 14 &= 1 \\
   x &= -13 \\
   \end{align*} \]
\[ (-13, 4, 6) \]
27. Identify the variables and set up the system of equations for the following problem. Do not solve.

A clothing company borrows $775,000. Some of the money is borrowed at 8%, some at 9%, and some at 10% simple interest. How much is borrowed at each rate if the total annual interest is $67,500 and the amount borrowed at 8% is four times the amount borrowed at 10%?

\[
x + y + z = 775,000
0.08x + 0.09y + 0.10z = 67,500
x = 4z
\]

Calculator Portion of Exam

28. The endpoints of the diameter of a circle are (-5,1) and (7,6).
   a. Find the center of the circle.

\[
\left( \frac{-5+7}{2}, \frac{1+6}{2} \right) = (1, \frac{7}{2}) = (1, 3.5)
\]

b. Find the radius of the circle.

\[
r = \sqrt{(-5-1)^2 + (1-3.5)^2} = \sqrt{36 + 6.25} = 6.5
\]

c. Find the standard form of the equation of the circle.

\[
(x-1)^2 + (y-3.5)^2 = 6.5^2
\]

29. A high school had an enrollment of 600 students in 1985. During the next 20 years, the enrollment increased by approximately 40 students per year.
   a. Write a linear equation giving the enrollment N in terms of the year t. (Let t=5 correspond to the year 1985.)

\[
m = \frac{40}{5} = 8 \quad \text{and} \quad t = 0 \Rightarrow 1980
\]

\[
y - y_1 = m (x-x_1)
\]

\[
y - 600 = 40 (x - 5)
\]

\[
y - 600 = 40x - 200
\]

\[
N = 40t + 400
\]

b. If this constant rate of growth continues, predict the enrollment in the year 2010.

\[
2010 : t = 30 \quad N = 40(30) + 400 = 1600
\]

30. A rock is dropped from the top of a 200-foot cliff that overlooks the ocean. How long will it take for the rock to hit the water?

\[
S = -16t^2 + V_0t + S_0
\]

\[
0 = -16t^2 + 0t + 200
\]

\[
0 = -16t^2 + 200
\]

\[
16t^2 = 200
\]

\[
t^2 = 12.5
\]

\[
t = \pm 3.54 \text{ seconds}
\]
31. A manufacturer of chairs has daily production costs (in dollars per chair) of $C(x) = 0.3x^2 - 12x + 5400$ where $x$ is the number of chairs produced. How many chairs should be produced each day to yield a minimum cost per unit?

\[ x = \frac{-b}{2a} = \frac{-(-12)}{2(0.3)} = \frac{12}{0.6} = 20 \text{ chairs} \]

32. Find the number of units that produces a maximum revenue. The revenue $R$ is measured in dollars and $x$ is the number of units produced.

\[ R = 50x - 0.0002x^2 \]

\[ x = \frac{-b}{2a} = \frac{-50}{2(-0.0002)} = \frac{-50}{-0.0004} = 125,000 \text{ units} \]

33. Solve the following for $x$:
   a. $\ln x - \ln(x + 1) = 3$
      \[ \ln \frac{x}{x+1} = 3 \]
      \[ \frac{\ln x}{\ln(x+1)} = e^3 \]
      \[ (x+1)(\frac{x}{x+1}) = 20.09 \]
      \[ x = 20.09(x+1) \]
      \[ x = 20.09x + 20.09 \]
      \[ -19.09x = 20.09 \]
      \[ x = \frac{20.09}{19.09} = 1.05 \]
      \[ x \approx 1.05 \text{ or No solution} \]

   b. $b/310 = \frac{(600(1+e^{2x})}{(1+e^{2x})} - \frac{600}{1+e^{2x}})
      \[ (1+e^{2x}) = 600 \]
      \[ 1+e^{2x} = 1.9355 \]
      \[ e^{2x} = 0.9355 \]
      \[ \ln e^{2x} = \ln 0.9355 \]
      \[ 2x = -0.0667 \]
      \[ x = -0.0333 \]

   c. $5 + 30e^{0.47x} = 10$
      \[ 30e^{0.47x} = 5 \]
      \[ e^{0.47x} = \frac{1}{6} \]
      \[ \ln e^{0.47x} = \ln \frac{1}{6} \]
      \[ 0.47x = \ln \frac{1}{6} \]
      \[ x = \frac{\ln \frac{1}{6}}{0.47} \]
      \[ x \approx -3.812 \]

34. Suppose a certain amount of money is invested at 11% interest compounded continuously, and suppose the balance in 10 years has grown to $19,205.

   a. Find the initial investment.
      \[ A = Pe^{rt} \]
      \[ 19,205 = Pe^{(0.11 \cdot 10)} \]
      \[ \frac{19205}{e^{(0.11 \cdot 10)}} = P \]
      \[ P \approx 6,392.79 \]

   b. Find the time to double.
      \[ A = Pe^{0.11t} \]
      \[ \ln 2 = \ln e^{0.11t} \]
      \[ \ln 2 = 0.11t \]
      \[ \frac{\ln 2}{0.11} = t \]
      \[ t = 6.3 \text{ years} \]
35. Paul swallows 18 grams of radioactive dye. Three weeks later, only 8 grams remain in his system. How much will remain in his system 10 weeks later?

\[ y = y_0 e^{-kt} \]
\[ 8 = 18 e^{-k(3)} \]
\[ \frac{4}{9} = e^{-3k} \]
\[ \ln \frac{4}{9} = \ln e^{-3k} \]
\[ -0.8109 = -3k \]
\[ k = -0.2703 \]
\[ y = 18 e^{-0.2703(10)} \]
\[ y = 1.2 \text{ grams} \]

36. Certain bacteria grow according to the model \( y = Ae^{bx} \) where \( y \) denotes the number of bacteria after \( x \) hours have elapsed. If the initial population of the bacteria is 15,000 and after four hours the population is 25,000, find

a. the value of \( A \).

\[ A = 15,000 \]

b. the value of \( b \).

\[ \frac{25000}{15000} = e^{4b} \]
\[ 1.6667 = e^{4b} \]
\[ \ln 1.6667 = \ln e^{4b} \]
\[ 0.5108 = 4b \]
\[ b = 0.1277 \]

c. the number of bacteria expected after seven hours.

\[ y = 15000 e^{0.1277(7)} \]
\[ y = 36,671.3 \]

37. Students in a 7th grade class were given an exam. During the next two years, the students were retested several times. The average score \( g \) can be approximated by the model \( g(t) = 93 - 20 \log_{10}(t + 1) \), where \( t \) is the time in months.

a. What was the average score on the original exam?

\[ g(0) = 93 - 20 \log (0+1) \]
\[ g(0) = 93 \]

b. After how many months did the average score drop below 50?

\[ 50 = 93 - 20 \log (t+1) \]
\[ -43 = -20 \log (t+1) \]
\[ 2.15 = \log (t+1) \]
\[ 10^{2.15} = 10^{\log (t+1)} \]
\[ 141.25 = t+1 \]
\[ 140.25 = t \]