Putnam Problem(s) of the Week

October 21, 2016

1. What is the largest integer \( n \) for which \( n^3 + 100 \) is divisible by \( n + 10 \)?

**Solution:** By the division algorithm, \( n^3 + 100 = (n + 10)(n^2 - 10n + 100) - 900 \), so

\[
\frac{n^3 + 100}{n + 10} = n^2 - 10n + 100 - \frac{900}{n + 10}.
\]

If \( n^3 + 100 \) is to be divisible by \( n + 10 \), then \( 900/(n + 10) \) must be an integer. Thus, the largest possible \( n \) for which this is true is \( n = 890 \).

2. Show that for every positive integer \( n \),

\[
\left( \frac{2n - 1}{e} \right)^{(2n-1)/2} < 1 \cdot 3 \cdot 5 \cdots (2n - 1) < \left( \frac{2n + 1}{e} \right)^{(2n+1)/2}.
\]

**Solution:** Take logarithms! By estimating the area under the graph of \( \ln x \) using upper and lower rectangles of width 2, we get

\[
\int_1^{2n-1} \ln x \, dx \leq 2(\ln(3) + \cdots + \ln(2n - 1))
\]

\[
\leq \int_3^{2n+1} \ln x \, dx.
\]
Since $\int \ln x \, dx = x \ln x - x + C$, we have, upon exponentiating and taking square roots,

$$\left( \frac{2n - 1}{e} \right)^{2n-1} < (2n - 1)^{\frac{2n-1}{2}} e^{-n+1}$$

$$\leq 1 \cdot 3 \cdots (2n - 1)$$

$$\leq (2n + 1)^{\frac{2n+1}{2}} e^{-n+1} \frac{e^{-n+1}}{3^{3/2}}$$

$$< \left( \frac{2n + 1}{e} \right)^{\frac{2n+1}{2}},$$

using the fact that $1 < e < 3$. 

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