Putnam Problem(s) of the Week

October 28, 2016

1. Consider a set $S$ and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

**Solution:** The hypothesis implies $((b * a) * b) * (b * a) = b$ for all $a, b \in S$ (by replacing $a$ by $b * a$), and hence $a * (b * a) = b$ for all $a, b \in S$ (using $(b * a) * b = a$).

2. Prove that there exist infinitely many integers $n$ such that $n, n + 1, n + 2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

**Solution:** First solution: Let $a$ be an even integer such that $a^2 + 1$ is not prime. (For example, choose $a \equiv 2 \pmod{5}$, so that $a^2 + 1$ is divisible by 5.) Then we can write $a^2 + 1$ as a difference of squares $x^2 - b^2$, by factoring $a^2 + 1$ as $rs$ with $r \geq s > 1$, and setting $x = (r + s)/2, b = (r - s)/2$. Finally, put $n = x^2 - 1$, so that $n = a^2 + b^2, n + 1 = x^2, n + 2 = x^2 + 1$.

Second solution: It is well-known that the equation $x^2 - 2y^2 = 1$ has infinitely many solutions (the so-called “Pell” equation). Thus setting $n = 2y^2$ (so that $n = y^2 + y^2, n + 1 = x^2 + 0^2, n + 2 = x^2 + 1^2)$ yields infinitely many $n$ with the desired property.

Third solution: As in the first solution, it suffices to exhibit $x$ such that $x^2 - 1$ is the sum of two squares. We will take $x = 3^{2^n}$, and show that $x^2 - 1$ is the sum of two squares by induction on $n$: if $3^{2^n} - 1 = a^2 + b^2$, then

$$
(3^{2^{n+1}} - 1) = (3^{2^n} - 1)(3^{2^n} + 1)
= (3^{2^{n-1}}a + b)^2 + (a - 3^{2^{n-1}}b)^2.
$$