1. Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

**Solution:** Yes, it does follow. Let $P$ be any point in the plane. Let $ABCD$ be any square with center $P$. Let $E, F, G, H$ be the midpoints of the segments $AB, BC, CD, DA$, respectively. The function $f$ must satisfy the equations

\[
\begin{align*}
0 &= f(A) + f(B) + f(C) + f(D) \\
0 &= f(E) + f(F) + f(G) + f(H) \\
0 &= f(A) + f(E) + f(P) + f(H) \\
0 &= f(B) + f(F) + f(P) + f(E) \\
0 &= f(C) + f(G) + f(P) + f(F) \\
0 &= f(D) + f(H) + f(P) + f(G).
\end{align*}
\]

If we add the last four equations, then subtract the first equation and twice the second equation, we obtain $0 = 4f(P)$, whence $f(P) = 0$.

**Remark.** Problem 1 of the 1996 Romanian IMO team selection exam asks the same question with squares replaced by regular polygons of any (fixed) number of vertices.

2. Given a positive integer $n$, what is the largest $k$ such that the numbers $1, 2, \ldots, n$ can be put into $k$ boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest $k$ is at least 3.]
Solution: The largest such $k$ is $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$. For $n$ even, this value is achieved by the partition

$$\{1, n\}, \{2, n - 1\}, \ldots;$$

for $n$ odd, it is achieved by the partition

$$\{n\}, \{1, n - 1\}, \{2, n - 2\}, \ldots.$$

One way to see that this is optimal is to note that the common sum can never be less than $n$, since $n$ itself belongs to one of the boxes. This implies that $k \leq (1 + \cdots + n)/n = (n+1)/2$. Another argument is that if $k > (n+1)/2$, then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.